Element and Material Formulations
An Overview

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Contents

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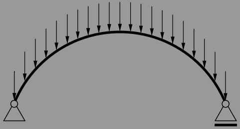
1. Nature

2. Physical model



3. Mathematical m.

$$\left.\begin{aligned} M'' - \tfrac{N}{R} + p_r &= 0 \\ \tfrac{M'}{R} + N' + p_\varphi &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} N &= EA\alpha - EI\,\tfrac{\omega}{R} \\ M &= EI\omega \end{aligned}\right\}$$

$$\left.\begin{aligned} \alpha &= \tfrac{u'}{R} + \tfrac{w}{R} \\ \omega &= -\tfrac{w''}{R^2} \end{aligned}\right\}$$

Idealization

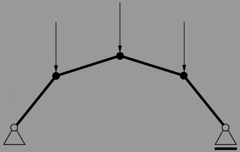
Theory

Discretization

6. Result

?

5. FE model



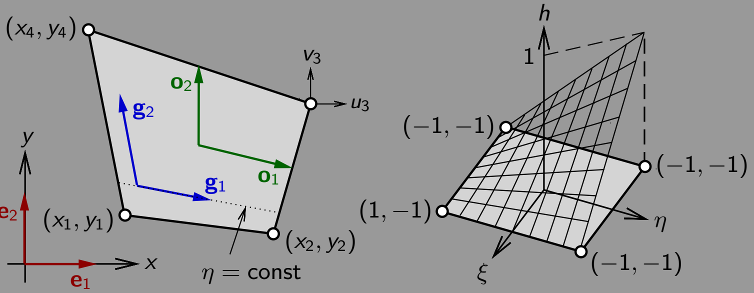
4. Numerical model

$$\begin{pmatrix} * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \\ p_{10} \end{pmatrix}$$

Solver

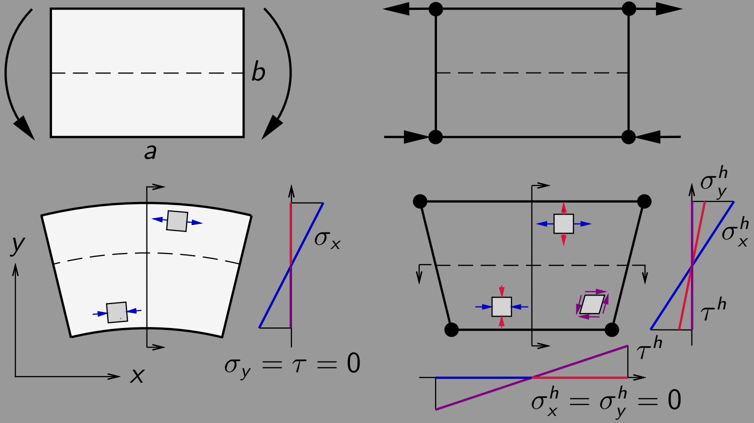
Element technology

A 4 node plane strain continuum element QC8
(Turner & Clough 1956)



$$\begin{bmatrix} u^h \\ v^h \end{bmatrix} = \sum_{n=1}^{4} h_n(\xi, \eta) \begin{bmatrix} u_n \\ v_n \end{bmatrix}$$

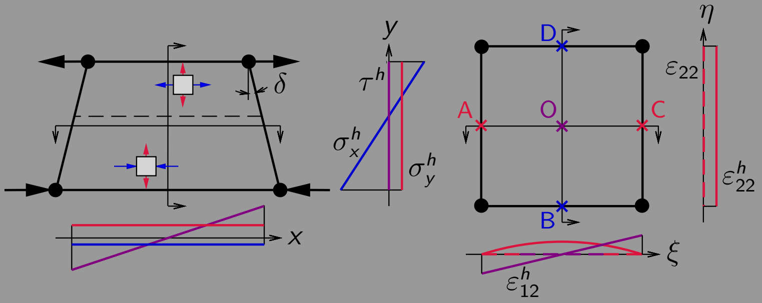
Rectangular QC4 element – pure bending



Shear and Poisson locking: $\frac{U^h}{U} = \frac{1-\nu}{2}\left(\frac{a}{b}\right)^2 + \frac{(1-\nu)^2}{1-2\nu}$

Trapezoidal QC4 element – pure bending

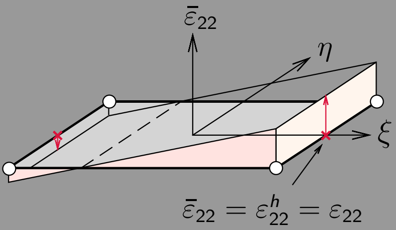
Cartesian stresses/strains ($\nu = 0$) *Natural* strains

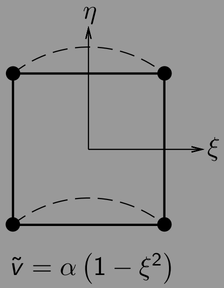


$$\frac{U^h}{U} = 1 + \frac{4}{5}\,\delta^2 + 3\left(\frac{a}{b}\right)^4 \delta^2 + \frac{1}{2}\left(\frac{a}{b}\right)^2 \left(1 - \delta^2\right) + O(\delta^4)$$

Assumed natural strains – enhanced assumed strains
(Wilson, Taylor, Hughes, Bathe, Simo, Ramm et al. 1969-1993)

$$\begin{aligned}
\varepsilon_{11}^h &= \gamma_1 + \gamma_4\,\eta && + \alpha_1\,\xi && + \alpha_5\,\xi\eta \\
\varepsilon_{22}^h &= \gamma_2 + \gamma_5\,\xi && + && + \alpha_2\,\eta + \alpha_6\,\xi\eta \\
\varepsilon_{12}^h &= \underbrace{\gamma_3}_{[\bar{\varepsilon}] = \{\mathbf{P}\}\,[\gamma]} && + && \underbrace{\alpha_3\,\xi + \alpha_4\,\eta + \alpha_7\,\xi\eta}_{[\tilde{\varepsilon}] = \{\mathbf{M}\}\,[\alpha]}
\end{aligned}$$





Assumed natural strains – enhanced assumed strains
(Wilson, Taylor, Hughes, Bathe, Simo, Ramm et al. 1969-1993)

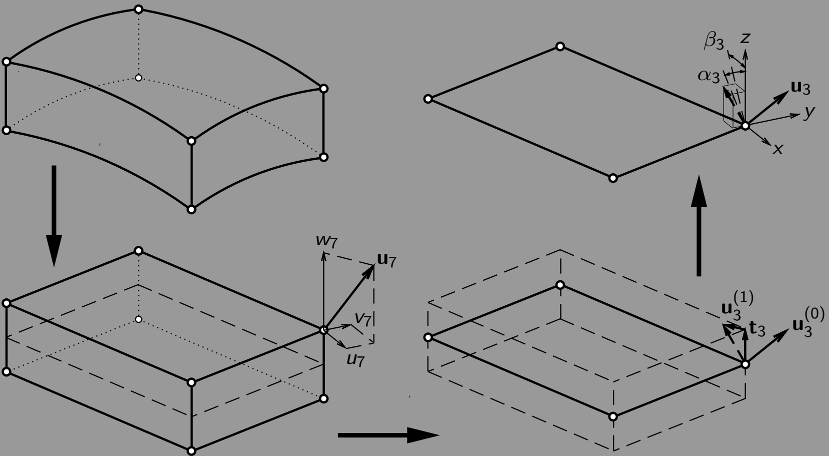
$$\begin{aligned}
\varepsilon_{11}^h &= \gamma_1 + \gamma_4\,\eta + \alpha_1\,\xi + \alpha_5\,\xi\eta \\
\varepsilon_{22}^h &= \gamma_2 + \gamma_5\,\xi + \alpha_2\,\eta + \alpha_6\,\xi\eta \\
\varepsilon_{12}^h &= \underbrace{\gamma_3}_{[\bar{\varepsilon}] = \{\mathbf{P}\}[\gamma]} + \underbrace{\alpha_3\,\xi + \alpha_4\,\eta + \alpha_7\,\xi\eta}_{[\tilde{\varepsilon}] = \{\mathbf{M}\}[\alpha]}
\end{aligned}$$

Hybrid stress (Pian & Sumihara 1984)

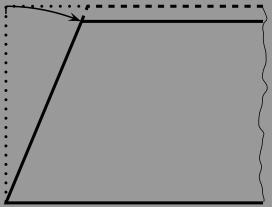
$$\begin{aligned}
\sigma_{11}^h &= \beta_1 + \beta_4\,\eta \\
\sigma_{22}^h &= \beta_2 + \beta_5\,\xi \\
\sigma_{12}^h &= \beta_3 \\
[\boldsymbol{\sigma}^h] &= \{\mathbf{P}\}[\beta]
\end{aligned}$$

$$\iint [\tilde{\varepsilon}]^T [\boldsymbol{\sigma}] \sqrt{g_o}\, d\xi d\eta = 0$$

From a continuum to a shell element (degeneration)

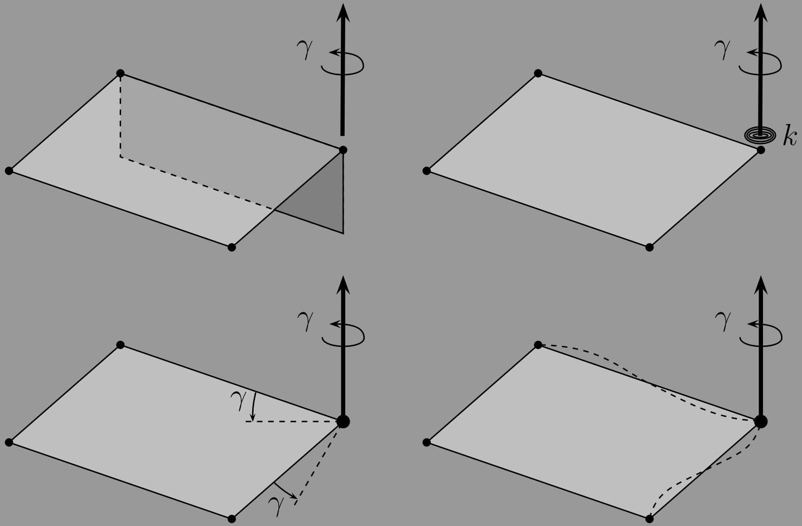


Reissner-Mindlin kinematics



- Not consistent with continuum mechanics
- No drill rotation

What about the drill rotation?



Mixed-hybrid shell strain/stress interpolation
(Klinkel, Gruttmann & Wagner 2008)

$$[\boldsymbol{\varepsilon}^h] = \{\mathbf{P}\}\,[\gamma^0] + \{\mathbf{M}\}\,[\boldsymbol{\alpha}^0], \qquad [\mathbf{n}^h] = \{\mathbf{P}\}\,[\boldsymbol{\beta}^0]$$
$$\varepsilon^h_{33} = \alpha^0_8$$
$$[\boldsymbol{\kappa}^h] = \{\mathbf{P}\}\,[\gamma^1] + \{\mathbf{M}\}\,[\boldsymbol{\alpha}^1], \qquad [\mathbf{m}^h] = \{\mathbf{P}\}\,[\boldsymbol{\beta}^1]$$
$$\kappa^h_{33} = \alpha^1_8$$

$$\{\mathbf{P}\} \sim \begin{Bmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & 0 & 0 \end{Bmatrix}, \qquad \{\mathbf{M}\} \sim \begin{Bmatrix} \xi & 0 & 0 & 0 & \xi\eta & 0 & 0 \\ 0 & \eta & 0 & 0 & 0 & \xi\eta & 0 \\ 0 & 0 & \xi & \eta & 0 & 0 & \xi\eta \end{Bmatrix}$$

Finite strain shell element formulations

ICONA (QMHS4)

$$\begin{cases} [\mathbf{x}_n]^{i+1} = [\mathbf{x}_n]^i + [\Delta \mathbf{x}_n] \\ [\omega_n]^{i+1} = [\omega_n]^i + [\Delta \omega_n] \end{cases}$$

$$\begin{cases} [\mathbf{x}]^h = \sum h_n [\mathbf{x}_n] \\ [\mathbf{Q} \cdot \mathbf{t}]^h = \sum h_n [\mathbf{Q}_n \cdot \mathbf{t}_n] \end{cases}$$

$$\|[\mathbf{Q} \cdot \mathbf{t}]^h\| \approx 1$$

Abaqus (S4)

$$\begin{cases} [\mathbf{x}_n]^{i+1} = [\mathbf{x_n}]^i + [\Delta \mathbf{x}_n] \\ \{\mathbf{Q}_n\}^{i+1} = \{\Delta \mathbf{Q}_n\} \cdot \{\mathbf{Q}_n\}^i \end{cases}$$

$$\begin{cases} [\mathbf{x}]^h = \sum h_n [\mathbf{x}_n] \\ [\Delta \omega]^h = \sum h_n [\Delta \omega_n] \end{cases}$$

$$\|[\mathbf{Q} \cdot \mathbf{t}]^h\| \equiv 1$$

Results 1

Pinched hemisphere (Abaqus S4)
(MacNeal & Harder 1985)

64×16 elements

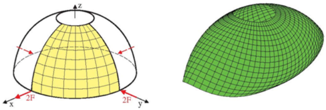
$E = 6.825 \cdot 10^7$

$\nu = 0.3$

$R = 10$

$t = 0.04$

$F = 200$



Ink.	$\Delta\lambda$	λ	Iter.
1	0.063	0.063	5
2	0.063	0.125	8
3	0.063	0.188	8
4	0.063	0.250	8
5	0.063	0.313	7
6	0.063	0.375	6
7	0.063	0.438	6
8	0.063	0.500	5
9	0.063	0.563	5
10	0.063	0.625	3
11	0.063	0.688	3
12	0.094	0.781	6
13	0.094	0.875	6
14	0.094	0.969	5
15	0.031	1.000	3
			84

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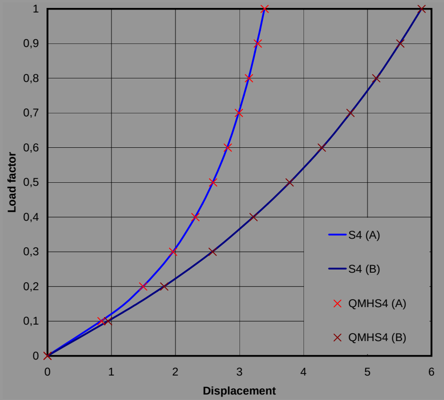
Pinched hemisphere (ICONA QMHS4)



Ink.	$\Delta\lambda$	λ	Iter.
1	0.500	0.500	6
2	0.500	1.000	9
			15

Large strains

- Isotropic linear elasticity + Mises plasticity (*F.-J. Falkner*, UIBK)

Moderate strains

- Anisotropic Hencky elasticity + Hoffmann-Hill plasticity (*R. Traxl*, UIBK)

Small strains

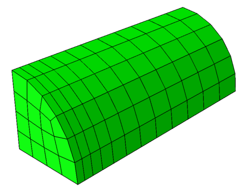
- Anisotropic linear elasticity + Hoffmann-Hill plasticity + multisurface damage criterion (*U. Hofer*, UIBK & INTALES)
- Two-phase composite model including plasticity and damage (*E. Eidelpes*, UIBK)

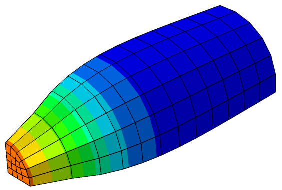
Finite strain elasto-plasticity models

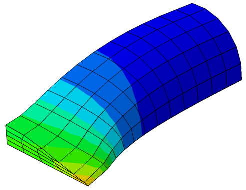
	FeLab/ICONA (TL)	Abaqus (UL)
Isotropic	$\mathbf{F} = \mathbf{F}^e \cdot \mathbf{F}^p$	$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p, \quad \mathbf{d} = \dot{\varepsilon}$
Anisotropic	$\hat{\varepsilon} = \hat{\varepsilon}^e + \hat{\varepsilon}^p$	$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p, \quad \mathbf{d} \approx \dot{\varepsilon}$
	$\hat{\tau} = \mathsf{C} : \hat{\varepsilon}$	$\dot{\tau} = \mathsf{C} : \mathbf{d}$
	$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}$	$\mathbf{d} = \mathrm{sym}\left(\nabla \otimes \mathbf{v}\right)$
	$\hat{\varepsilon} = \ln \mathbf{U}$	$\varepsilon = \ln \mathbf{V}$
	$= \mathbf{R}^T \cdot \varepsilon \cdot \mathbf{R}$	

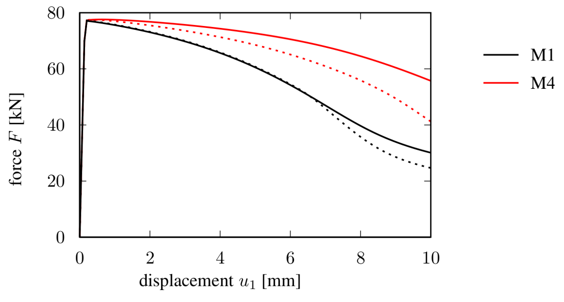
Necking of a rod

(Eidel & Gruttmann 2003)









FeLab/ICONA (QMHS4)

- TL formulation
  - (+) Theoretically sound formulation
  - (+) Compatible with recent developments
- Integral constitutive laws (large strain anisotropy)
  - (−) Highly complex formulations
  - (−) Approximations and/or heuristic assumptions
  - (−) Moderate computational performance
- Mixed-hybrid formulation
  - (+) Extemely large load steps
  - (−) Increased effort for element operations
- 5/6 DOF formulation
  - (+) No algorithmic parameters
  - (+) Reduced number of global DOFs
  - (−) Additional effort for preprocessing
- 'Continuum consistent' discretization
  - (+) Thickness strains and higher-order shear strains
  - (−) Lower corse mesh accuracy

Abaqus (S4)

- UL formulation
  - (o) No proper variational basis
  - (o) Rarely covered in the recent literature
- Rate-form constitutive laws
  - (+) Relatively simple formulations
  - (−) Problems arise for large strains ($\varepsilon > 20\,\%$) at anisotropy
  - (+) High computational performance
- Reduced integration
  - (+) Highly efficient element operations
  - (−) Involves additional algorithmic parameters
- 6 DOF formulation
  - (−) Involves algorithmic parameters
  - (−) Enhanced number of global DOFs
  - (+) No extra effort for preprocessing
- Shell theory consistent formulation
  - (−) No thickness strains
  - (+) Higher corse mesh accuracy